Analysis of leader election process in proof of stake consensus model

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Abstract. We consider leader election process in proof of stake (PoS) consensus model.

1. The Model

- We consider N nodes where node i has stake $s_i \in (0,1]$ such that $\sum_{i=1}^{N} s_i = 1$.
- We assume that at time t all nodes participate in a "lottery" and we model outcome of this lottery for node i by the variable $\sigma_i(t) \in \{0,1\}$, where 0 and 1 corresponds to, respectively, "loosing" and "winning" in this lottery.
- We assume that for node i the probability of winning $\mathrm{P}(\sigma_i(t)=1) = s_i$.
- For times $t \in [\tau]$, where we used $[\tau] = \{1,\ldots,\tau\}$, the sequence $\sigma_i(1),\ldots,\sigma_i(\tau)$ is the record of lottery events for node i.
- We assume that the outcome of lottery of node i at time t, $\sigma_i(t)$, is either observed or not observed. The latter is modelled by variable $\eta_i(t) \in \{0,1\}$, where 0/1 corresponds to "not observed"/"observed".
- We assume that for node i the probability of observing $\mathrm{P}(\eta_i(t)=1) = q$.
- For times $t \in [\tau]$ the sequence $\eta_i(1),\ldots,\eta_i(\tau)$ is the record of observation events for node i.

2. Probability distributions

- For node i we are interested in the probability distribution of random variables $n = \sum_{t=1}^{\tau} \eta_i(t)\,\sigma_i(t)$, i.e. the number of lottery wins observed, and $m = \sum_{t=1}^{\tau} \eta_i(t)$, i.e. the number of observations. The latter two, for $m > 1$, allow us to construct the statistical estimator n/m of the stake s_i.
- For any node i with the stake $s_i = s$ the joint distribution of n and m is given by

$$\mathrm{P}\left(n,m|\tau\right) = \sum_{\sigma(1)}\cdots\sum_{\sigma(\tau)}\sum_{\eta(1)}\cdots\sum_{\eta(\tau)}\prod_{t=1}^{\tau}\mathrm{P}(\sigma(t))\,\mathrm{P}(\eta(t)) \times \delta_{n;\sum_{t=1}^{\tau}\eta_i(t)\,\sigma_i(t)}\delta_{m;\sum_{t=1}^{\tau}\eta_i(t)}, \tag{1}$$

where $\delta_{x;y}$ is the Kronecker delta function, i.e. for $x,y \in \mathbb{Z}$ the $\delta_{x;y} = 1$ if $x = y$ and $\delta_{x;y} = 0$ if $x \neq y$.

- The sums in above can be computed, by using the Fourier representation $\delta_{x;y} = \int_{-\pi}^{\pi} \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{\mathrm{i}\omega(x-y)}$ in and exploiting properties a binomial distribution, giving us

$$\mathrm{P}\left(n, m|\tau\right) = \mathrm{P}\left(n|m\right)\mathrm{P}\left(m|\tau\right), \tag{2}$$

where $\mathrm{P}\left(m|\tau\right)$ is the binomial distribution

$$\mathrm{P}\left(m|\tau\right) = \binom{\tau}{m} q^m \left(1-q\right)^{\tau-m} \tag{3}$$

with the parameter q such that $q\,\tau$ is the *average* number of observations‡, and

$$\mathrm{P}\left(n|m\right) = \binom{m}{n} s^n \left(1-s\right)^{m-n} \tag{4}$$

is the binomial distribution with parameter s such that $s\,m$ is the average number of observed lottery wins.

- The distribution (2) can be used to construct

$$\begin{aligned}\mathrm{P}\left(n, m|\tau, m\geq 1\right) &= \frac{\mathrm{P}\left(n, m|\tau\right)\mathbb{1}[m\geq 1]}{\sum_{\tilde{m}=0}^{\tau}\sum_{\tilde{n}=0}^{\tilde{m}}\mathrm{P}\left(\tilde{n}, \tilde{m}|\tau\right)\mathbb{1}[\tilde{m}\geq 1]}\\ &= \frac{\mathrm{P}\left(m|\tau\right)\mathrm{P}\left(n|m\right)\mathbb{1}[m\geq 1]}{1-\left(1-q\right)^{\tau}},\end{aligned} \tag{5}$$

i.e. the probability of n and m given that $m \geq 1$.

- We note that

$$\begin{aligned}\sum_{m=0}^{\tau}\sum_{n=0}^{m}\mathrm{P}\left(n, m|\tau, m\geq 1\right)\frac{n}{m} &= \sum_{m=1}^{\tau}\frac{\mathrm{P}\left(m|\tau\right)}{1-\left(1-q\right)^{\tau}}\frac{sm}{m}\\ &= s\end{aligned} \tag{6}$$

and hence given that $\sum_{t=1}^{\tau}\eta_i(t) \geq 1$, i.e. at least one observation event happened, the fraction $\sum_{t=1}^{\tau}\eta_i(t)\,\sigma_i(t)/\sum_{t=1}^{\tau}\eta_i(t)$ is *unbiased* estimator of the stake s.

3. Concentration inequalities

- For $\epsilon > 0$ we are interested in the probability of the event $s-\epsilon \leq n/m \leq s+\epsilon$, where $m \geq 1$. The latter is given by

$$\begin{aligned}&\mathrm{P}\left(\left(s-\epsilon\right)m \leq n \leq \left(s+\epsilon\right)m|m\geq 1\right)\\ &\quad = 1-\mathrm{P}\left(n\leq\lfloor\left(s-\epsilon\right)m\rfloor|m\geq 1\right)\\ &\qquad -\mathrm{P}\left(n\geq\lfloor\left(s+\epsilon\right)m\rfloor+1|m\geq 1\right),\end{aligned} \tag{7}$$

where in above we assumed that $(s\pm\epsilon)\,m \notin \mathbb{N}$. To estimate above we first consider the probability

$$\begin{aligned}&\mathrm{P}\left(n\geq\lfloor\left(s+\epsilon\right)m\rfloor+1|m\geq 1\right)\\ &\quad = \sum_{m=0}^{\tau}\sum_{n=0}^{m}\mathrm{P}\left(n, m|\tau, m\geq 1\right)\mathbb{1}[n\geq\lfloor\left(s+\epsilon\right)m\rfloor+1]\end{aligned}$$

‡ We note that *mode* of the binomial distribution is (approx.) at $q\tau$ and hence the *typical* number of observations will be also (approx.) $q\tau$. Furthermore, for $A > q$, where $A \in (0,1)$, the $\mathrm{P}(m \geq \lfloor A\,\tau\rfloor) \leq \mathrm{e}^{-\tau\mathrm{D}(A||q)}$, where $\mathrm{D}\left(A||q\right)$ is the Kullback–Leibler (KL) 'distance' [1]. Using $\mathrm{D}\left(A||q\right) \geq 2(A-q)^2$ and $q+\epsilon$ in the latter we have $\mathrm{P}(m \geq \lfloor(q+\epsilon)\,\tau\rfloor) \leq \mathrm{e}^{-2\tau\epsilon^2}$.

$$= \frac{\sum_{m=1}^{\tau} \mathrm{P}\left(m|\tau\right) \sum_{n=\lfloor (s+\epsilon)\, m \rfloor + 1}^{m} \mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}$$
$$\leq \frac{\sum_{m=1}^{\tau} \mathrm{P}\left(m|\tau\right) \left\{ \mathbb{1}[m=1]s + \mathbb{1}[m \geq 2] \mathrm{e}^{-m\, \mathrm{D}(s+\epsilon||s)} \right\}}{1-(1-q)^{\tau}}$$
$$= \frac{\mathrm{P}\left(1|\tau\right)\, s + \sum_{m=2}^{\tau} \mathrm{P}\left(m|\tau\right) \mathrm{e}^{-m\, \mathrm{D}(s+\epsilon||s)}}{1-(1-q)^{\tau}} \tag{8}$$

where we used upper bound on the upper tail of the binomial distribution derived in the Appendix A, and hence computing the sum in the above we obtain the following inequality

$$\mathrm{P}\left(n \geq \lfloor (s+\epsilon)\, m \rfloor + 1 | m \geq 1\right)$$
$$\leq \frac{q(1-q)^{\tau-1} \left[s - \mathrm{e}^{-\mathrm{D}(s+\epsilon||s)} \right] - (1-q)^{\tau}}{1-(1-q)^{\tau}}$$
$$+ \frac{\left[1-q+q\, \mathrm{e}^{-\mathrm{D}(s+\epsilon||s)} \right]^{\tau}}{1-(1-q)^{\tau}}. \tag{9}$$

Secondly, we consider

$$\mathrm{P}\left(n \leq \lfloor (s-\epsilon)\, m \rfloor | m \geq 1\right)$$
$$= \frac{\sum_{m=1}^{\tau} \mathrm{P}\left(m|\tau\right) \sum_{n=0}^{\lfloor (s-\epsilon)\, m \rfloor} \mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}$$
$$= \frac{\sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \sum_{n=0}^{\lfloor (s-\epsilon)\, m \rfloor} \mathrm{P}\left(n|m\right) - (1-q)^{\tau}}{1-(1-q)^{\tau}}$$
$$= \frac{\sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[(s-\epsilon)\, m < 1] (1-s)^{m}}{1-(1-q)^{\tau}}$$
$$+ \frac{\sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[(s-\epsilon)\, m \geq 1] \sum_{n=0}^{\lfloor (s-\epsilon)\, m \rfloor} \mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}$$
$$- \frac{(1-q)^{\tau}}{1-(1-q)^{\tau}}. \tag{10}$$

Now the sum

$$\sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[(s-\epsilon)\, m < 1] (1-s)^{m}$$
$$\leq \sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) (1-s)^{m} = (1-qs)^{\tau} \tag{11}$$

and the sum

$$\sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[(s-\epsilon)\, m \geq 1] \sum_{n=0}^{\lfloor (s-\epsilon)\, m \rfloor} \mathrm{P}\left(n|m\right)$$
$$\leq \sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[(s-\epsilon)\, m \geq 1]\, \mathrm{e}^{-m\, \mathrm{D}(s-\epsilon||s)}$$
$$\leq \sum_{m=0}^{\tau} \mathrm{P}\left(m|\tau\right)\, \mathrm{e}^{-m\, \mathrm{D}(s-\epsilon||s)}$$
$$= \left[1-q+q\, \mathrm{e}^{-\mathrm{D}(s-\epsilon||s)} \right]^{\tau} \tag{12}$$

where in above we used results from the Appendix A, and hence

$$\begin{aligned}&\mathrm{P}\left(n \le \lfloor (s-\epsilon)\, m \rfloor | m \ge 1\right)\\ &\qquad \le \frac{(1-qs)^\tau + \left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s-\epsilon||s)}\right]^\tau - (1-q)^\tau}{1-(1-q)^\tau}.\end{aligned} \tag{13}$$

Using the inequalities (9) and (13) gives us

$$\begin{aligned}&\mathrm{P}\left(n \le \lfloor (s-\epsilon)\, m \rfloor | m \ge 1\right)\\ &\qquad + \mathrm{P}\left(n \ge \lfloor (s+\epsilon)\, m \rfloor + 1 | m \ge 1\right)\\ &\qquad \le \frac{(1-qs)^\tau + \left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s-\epsilon||s)}\right]^\tau - (1-q)^\tau}{1-(1-q)^\tau}\\ &\qquad\quad + \frac{q(1-q)^{\tau-1}\left[s - \mathrm{e}^{-\mathrm{D}(s+\epsilon||s)}\right] - (1-q)^\tau}{1-(1-q)^\tau}\\ &\qquad\quad + \frac{\left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s+\epsilon||s)}\right]^\tau}{1-(1-q)^\tau}\end{aligned} \tag{14}$$

and hence for (7) we obtain the following lower bound

$$\begin{aligned}&\mathrm{P}\left((s-\epsilon)\, m \le n \le (s+\epsilon)\, m | m \ge 1\right)\\ &\qquad \ge 1 - \frac{(1-qs)^\tau + \left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s-\epsilon||s)}\right]^\tau - (1-q)^\tau}{1-(1-q)^\tau}\\ &\qquad\quad - \frac{q(1-q)^{\tau-1}\left[s - \mathrm{e}^{-\mathrm{D}(s+\epsilon||s)}\right] - (1-q)^\tau}{1-(1-q)^\tau}\\ &\qquad\quad - \frac{\left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s+\epsilon||s)}\right]^\tau}{1-(1-q)^\tau}.\end{aligned} \tag{15}$$

To compute upper bound on (7) we first consider

$$\begin{aligned}&\mathrm{P}\left(n \ge \lfloor (s+\epsilon)\, m \rfloor + 1 | m \ge 1\right)\\ &\quad = \sum_{m=0}^{\tau}\sum_{n=0}^{m} \mathrm{P}\left(n, m | \tau, m \ge 1\right)\\ &\qquad \times \mathbb{1}[n \ge \lfloor (s+\epsilon)\, m \rfloor + 1]\\ &\quad = \sum_{m=1}^{\tau} \frac{\mathrm{P}\left(m|\tau\right)}{1-(1-q)^\tau}\\ &\qquad \times \sum_{n=0}^{m} \mathrm{P}\left(n|m\right) \mathbb{1}[n \ge \lfloor (s+\epsilon)\, m \rfloor + 1]\\ &\qquad\quad \ge \frac{\sum_{m=1}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[m=1] s}{1-(1-q)^\tau}\\ &\qquad\qquad + \frac{\sum_{m=1}^{\tau} \mathrm{P}\left(m|\tau\right) \mathbb{1}[m \ge 2] \mathrm{e}^{-m\,\mathrm{D}(s+\epsilon+1/2||s)}}{[1-(1-q)^\tau]\sqrt{8\tau \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\left(1 - \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\right)}}\\ &\qquad\qquad = \frac{q(1-q)^{\tau-1} s}{1-(1-q)^\tau}\end{aligned}$$

$$+\frac{\left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}\right]^{\tau}}{[1-(1-q)^{\tau}]\sqrt{8\tau\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\left(1-\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\right)}}$$
$$-\frac{(1-q)^{\tau}+q(1-q)^{\tau-1}\mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}}{[1-(1-q)^{\tau}]\sqrt{8\tau\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\left(1-\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\right)}} \quad (16)$$

and hence from above follows

$$\begin{aligned}
&\mathrm{P}\left(n\geq\lfloor(s+\epsilon)\,m\rfloor+1|m\geq 1\right)\\
&\quad\geq\frac{q(1-q)^{\tau-1}s}{1-(1-q)^{\tau}}\\
&\qquad+\frac{\left[1-q+q\,\mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}\right]^{\tau}}{[1-(1-q)^{\tau}]\sqrt{8\tau\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\left(1-\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\right)}}\\
&\qquad-\frac{(1-q)^{\tau}+q(1-q)^{\tau-1}\mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}}{[1-(1-q)^{\tau}]\sqrt{8\tau\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\left(1-\frac{\lfloor(s+\epsilon)\tau\rfloor+1}{\tau}\right)}}. \quad (17)
\end{aligned}$$

Now we consider

$$\begin{aligned}
&\mathrm{P}\left(n\leq\lfloor(s-\epsilon)\,m\rfloor|m\geq 1\right)\\
&\quad=\frac{\sum_{m=1}^{\tau}\mathrm{P}\left(m|\tau\right)\sum_{n=0}^{\lfloor(s-\epsilon)\,m\rfloor}\mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}\\
&\qquad=\mathbb{1}[q\tau<1]\frac{\sum_{m=1}^{\tau}\mathrm{P}\left(m|\tau\right)\sum_{n=0}^{\lfloor(s-\epsilon)\,m\rfloor}\mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}\\
&\qquad\quad+\mathbb{1}[q\tau\geq 1]\frac{\sum_{m=1}^{\tau}\mathrm{P}\left(m|\tau\right)\sum_{n=0}^{\lfloor(s-\epsilon)\,m\rfloor}\mathrm{P}\left(n|m\right)}{1-(1-q)^{\tau}}\\
&\qquad\quad\geq\mathbb{1}[q\tau<1]\frac{\mathrm{P}\left(1|\tau\right)\sum_{n=0}^{\lfloor(s-\epsilon)\,m\rfloor}\mathrm{P}\left(n|1\right)}{1-(1-q)^{\tau}}\\
&\qquad\quad\quad+\mathbb{1}[q\tau\geq 1]\frac{\mathrm{P}\left(\lfloor\tau q\rfloor|\tau\right)\sum_{n=0}^{\lfloor(s-\epsilon)\,\lfloor\tau q\rfloor\rfloor}\mathrm{P}\left(n|\lfloor\tau q\rfloor\right)}{1-(1-q)^{\tau}}\\
&\qquad\quad=\mathbb{1}[q\tau<1]\frac{\tau q(1-q)^{\tau-1}(1-s)}{1-(1-q)^{\tau}}\\
&\qquad\quad\quad+\mathbb{1}[q\tau\geq 1]\frac{\mathrm{P}\left(\lfloor\tau q\rfloor|\tau\right)}{1-(1-q)^{\tau}}\Bigg\{\mathbb{1}[(s-\epsilon)\,\lfloor\tau q\rfloor<1](1-s)^{\lfloor\tau q\rfloor}\\
&\qquad\quad\quad+\mathbb{1}[(s-\epsilon)\,\lfloor\tau q\rfloor\geq 1]\sum_{n=0}^{\lfloor(s-\epsilon)\,\lfloor\tau q\rfloor\rfloor}\mathrm{P}\left(n|\lfloor\tau q\rfloor\right)\Bigg\}\\
&\qquad\quad\geq\mathbb{1}[q\tau<1]\frac{\tau q(1-q)^{\tau-1}(1-s)}{1-(1-q)^{\tau}}\\
&\qquad\quad\quad+\mathbb{1}[q\tau\geq 1]\frac{1}{1-(1-q)^{\tau}}\frac{\mathrm{e}^{-\tau\,\mathrm{D}(\frac{\lfloor\tau q\rfloor}{\tau}||q)}}{\sqrt{8\tau\frac{\lfloor\tau q\rfloor}{\tau}\left(1-\frac{\lfloor\tau q\rfloor}{\tau}\right)}}\times\cdots
\end{aligned}$$

$$\cdots \times \left\{ \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor < 1](1-s)^{\lfloor \tau q \rfloor} + \cdots \right.$$
$$\left. \cdots + \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor \geq 1] \frac{\mathrm{e}^{-\lfloor \tau q \rfloor \, \mathrm{D}(\frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} || s)}}{\sqrt{8 \lfloor \tau q \rfloor \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} \left(1 - \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}\right)}} \right\} \tag{18}$$

and hence

$$\begin{aligned}
&\mathrm{P}\left(n \leq \lfloor (s-\epsilon)\, m \rfloor | m \geq 1\right) \\
&\quad \geq \mathbb{1}[q\tau < 1] \frac{\tau q (1-q)^{\tau-1}(1-s)}{1-(1-q)^{\tau}} \\
&\qquad + \mathbb{1}[q\tau \geq 1] \frac{1}{1-(1-q)^{\tau}} \frac{\mathrm{e}^{-\tau \, \mathrm{D}(\frac{\lfloor \tau q \rfloor}{\tau} || q)}}{\sqrt{8\tau \frac{\lfloor \tau q \rfloor}{\tau}\left(1 - \frac{\lfloor \tau q \rfloor}{\tau}\right)}} \times \cdots \\
&\qquad \cdots \times \left\{ \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor < 1](1-s)^{\lfloor \tau q \rfloor} + \cdots \right. \\
&\qquad \left. \cdots + \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor \geq 1] \frac{\mathrm{e}^{-\lfloor \tau q \rfloor \, \mathrm{D}(\frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} || s)}}{\sqrt{8 \lfloor \tau q \rfloor \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} \left(1 - \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}\right)}} \right\}
\end{aligned} \tag{19}$$

Using the bounds (17) and (19) gives the upper bound

$$\begin{aligned}
&\mathrm{P}\left((s-\epsilon)\, m \leq n \leq (s+\epsilon)\, m | m \geq 1\right) \\
&\leq 1 - \frac{q(1-q)^{\tau-1} s}{1-(1-q)^{\tau}} \\
&\qquad - \frac{\left[1 - q + q\, \mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}\right]^{\tau}}{[1-(1-q)^{\tau}] \sqrt{8\tau \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\left(1 - \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\right)}} \\
&\qquad + \frac{(1-q)^{\tau} + q(1-q)^{\tau-1} \mathrm{e}^{-\mathrm{D}(s+\epsilon+1/2||s)}}{[1-(1-q)^{\tau}] \sqrt{8\tau \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\left(1 - \frac{\lfloor (s+\epsilon)\tau \rfloor + 1}{\tau}\right)}} \\
&\qquad - \mathbb{1}[q\tau < 1] \frac{\tau q (1-q)^{\tau-1}(1-s)}{1-(1-q)^{\tau}} \\
&\qquad - \mathbb{1}[q\tau \geq 1] \frac{1}{1-(1-q)^{\tau}} \frac{\mathrm{e}^{-\tau \, \mathrm{D}(\frac{\lfloor \tau q \rfloor}{\tau} || q)}}{\sqrt{8\tau \frac{\lfloor \tau q \rfloor}{\tau}\left(1 - \frac{\lfloor \tau q \rfloor}{\tau}\right)}} \times \cdots \\
&\qquad \cdots \times \left\{ \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor < 1](1-s)^{\lfloor \tau q \rfloor} + \cdots \right. \\
&\qquad \left. \cdots + \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor \geq 1] \frac{\mathrm{e}^{-\lfloor \tau q \rfloor \, \mathrm{D}(\frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} || s)}}{\sqrt{8 \lfloor \tau q \rfloor \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor} \left(1 - \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}\right)}} \right\}
\end{aligned} \tag{20}$$

- If for $\epsilon > 0$ we are just interested in the probability of the event $n/m \geq s - \epsilon$, where $m \geq 1$, then the latter is given by

$$\begin{aligned}
&\mathrm{P}\left(n \geq \lfloor (s-\epsilon)\, m \rfloor + 1 | m \geq 1\right) \\
&\quad = 1 - \mathrm{P}\left(n \leq \lfloor (s-\epsilon)\, m \rfloor | m \geq 1\right) \\
&\quad \geq 1 - \frac{(1-q\,s)^\tau + \left[1 - q + q\,\mathrm{e}^{-\mathrm{D}(s-\epsilon||s)}\right]^\tau - (1-q)^\tau}{1-(1-q)^\tau} \\
&\quad \leq 1 - \mathbb{1}[q\tau < 1] \frac{\tau q (1-q)^{\tau-1}(1-s)}{1-(1-q)^\tau} \\
&\qquad\quad - \mathbb{1}[q\tau \geq 1] \frac{1}{1-(1-q)^\tau} \frac{\mathrm{e}^{-\tau\,\mathrm{D}(\frac{\lfloor \tau q \rfloor}{\tau}||q)}}{\sqrt{8\tau \frac{\lfloor \tau q \rfloor}{\tau}\left(1 - \frac{\lfloor \tau q \rfloor}{\tau}\right)}} \times \cdots \\
&\qquad\quad \cdots \times \Bigg\{ \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor < 1](1-s)^{\lfloor \tau q \rfloor} + \cdots \\
&\qquad\quad \cdots + \mathbb{1}[(s-\epsilon)\lfloor \tau q \rfloor \geq 1] \frac{\mathrm{e}^{-\lfloor \tau q \rfloor\,\mathrm{D}(\frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}||s)}}{\sqrt{8\lfloor \tau q \rfloor \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}\left(1 - \frac{\lfloor (s-\epsilon)\lfloor \tau q \rfloor \rfloor}{\lfloor \tau q \rfloor}\right)}} \Bigg\}
\end{aligned} \tag{21}$$

  where in above we used the inequalities (13) and (19) to obtain, respectively, the lower and upper bounds.

4. Analysis of asymptotic regime

- We are interested in the distribution of $\mu = m/\tau$ and $\nu = n/\tau$ random variables when τ is *very large*.
- For some test function $f(x, y)$ we consider the sum

$$\begin{aligned}
&\sum_{m=1}^{\tau}\sum_{n=0}^{m} \mathrm{P}\left(m|\tau\right)\mathrm{P}\left(n|m\right)\; f(m/\tau, n/\tau) \\
&\quad = \sum_{m=0}^{\tau}\sum_{n=0}^{m} \mathrm{P}\left(m|\tau\right)\mathrm{P}\left(n|m\right)\; f(m/\tau, n/\tau) - (1-q)^\tau f(0,0) \\
&\quad = \int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{P}_\tau(\mu,\nu)\, f(\mu,\nu) - (1-q)^\tau f(0,0) \\
&\quad = \int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \left[\mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau \delta(\mu)\,\delta(\nu)\right]\; f(\mu,\nu),
\end{aligned} \tag{22}$$

  where we defined the probability distribution

$$\mathrm{P}_\tau(\mu,\nu) = \sum_{m=0}^{\tau}\sum_{n=0}^{m} \mathrm{P}\left(m|\tau\right)\mathrm{P}\left(n|m\right)\; \delta(\mu - m/\tau)\,\delta(\nu - n/\tau). \tag{23}$$

  We note that for $\tau \to \infty$ the last term in above is *vanishing*. However, for *finite* τ we can define

$$\tilde{\mathrm{P}}_\tau(\mu,\nu) = \frac{\mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau \delta(\mu)\,\delta(\nu)}{1-(1-q)^\tau}. \tag{24}$$

- The averages of ν and μ are given by, respectively, the sums

$$\sum_{m=1}^{\tau}\sum_{n=0}^{m}\mathrm{P}\,(m|\tau)\,\mathrm{P}\,(n|m)\,n/\tau = \sum_{m=1}^{\tau}\mathrm{P}\,(m|\tau)\,s\,m/\tau$$
$$= \sum_{m=0}^{\tau}\mathrm{P}\,(m|\tau)\,s\,m/\tau = s\,q \qquad (25)$$

and

$$\sum_{m=1}^{\tau}\sum_{n=0}^{m}\mathrm{P}\,(m|\tau)\,\mathrm{P}\,(n|m)\,m/\tau = \sum_{m=1}^{\tau}\mathrm{P}\,(m|\tau)\,m/\tau$$
$$= \sum_{m=0}^{\tau}\mathrm{P}\,(m|\tau)\,m/\tau = q. \qquad (26)$$

- The distribution (23) can be computed in the limit $\tau\to\infty$ as follows. First, using the Fourier representation

$$\delta(\mu - m/\tau) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\mu}}{2\pi/\tau}\mathrm{e}^{\mathrm{i}\hat{\mu}\tau(\mu-m/\tau)} \qquad (27)$$

we compute the sums

$$\sum_{m=1}^{\tau}\mathrm{P}\,(m|\tau)\,\delta(\mu - m/\tau)$$
$$= \sum_{m=0}^{\tau}\mathrm{P}\,(m|\tau)\,\delta(\mu - m/\tau)$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\mu}}{2\pi/\tau}\mathrm{e}^{\mathrm{i}\hat{\mu}\mu\tau}\sum_{m=0}^{\tau}\mathrm{P}\,(m|\tau)\,\mathrm{e}^{-\mathrm{i}\hat{\mu}m}$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\mu}}{2\pi/\tau}\mathrm{e}^{\mathrm{i}\hat{\mu}\mu\tau}\left[1-q+q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}\right]^{\tau}$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\mu}}{2\pi/\tau}\mathrm{e}^{\tau\left[\mathrm{i}\hat{\mu}\mu+\log\left(1-q+q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}\right)\right]} \qquad (28)$$

and

$$\sum_{n=0}^{m}\mathrm{P}\,(n|m)\,\delta(\nu - n/\tau)$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\nu}}{2\pi/\tau}\mathrm{e}^{\mathrm{i}\hat{\nu}\nu\tau}\sum_{n=0}^{m}\mathrm{P}\,(n|m)\,\mathrm{e}^{-\mathrm{i}\hat{\nu}n}$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\nu}}{2\pi/\tau}\mathrm{e}^{\mathrm{i}\hat{\nu}\nu\tau}\left(1-s+s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}\right)^{m}$$
$$= \int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\nu}}{2\pi/\tau}\mathrm{e}^{\tau\left[\mathrm{i}\hat{\nu}\nu+\frac{m}{\tau}\log\left(1-s+s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}\right)\right]}. \qquad (29)$$

Now using above we have

$$\int_0^1\mathrm{d}\mu\int_0^{\mu}\mathrm{d}\nu\,\mathrm{P}_{\tau}(\mu,\nu)\,f(\mu,\nu)$$
$$= \int_0^1\mathrm{d}\mu\int_0^{\mu}\mathrm{d}\nu\int_{-\infty}^{\infty}\frac{\mathrm{d}\hat{\mu}}{2\pi/\tau}\mathrm{e}^{\tau\left[\mathrm{i}\hat{\mu}\mu+\log\left(1-q+q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}\right)\right]}$$

$$\times \int_{-\infty}^{\infty} \frac{\mathrm{d}\hat{\nu}}{2\pi/\tau} \mathrm{e}^{\tau\left[\mathrm{i}\hat{\nu}\nu+\mu\log\left(1-s+s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}\right)\right]} f(\mu,\nu)$$
$$= \frac{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu \int_{-\infty}^{\infty} \mathrm{d}\hat{\mu} \int_{-\infty}^{\infty} \mathrm{d}\hat{\nu}\, \mathrm{e}^{\tau\Psi[\mu,\nu,\hat{\mu},\hat{\nu}]} f(\mu,\nu)}{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu \int_{-\infty}^{\infty} \mathrm{d}\hat{\mu} \int_{-\infty}^{\infty} \mathrm{d}\hat{\nu}\, \mathrm{e}^{\tau\Psi[\mu,\nu,\hat{\mu},\hat{\nu}]}}, \tag{30}$$

where we defined the function

$$\Psi[\mu,\nu,\hat{\mu},\hat{\nu}] = \mathrm{i}\hat{\mu}\mu + \log\left(1-q+q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}\right) + \mathrm{i}\hat{\nu}\nu + \mu\log\left(1-s+s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}\right). \tag{31}$$

For $\tau \to \infty$ we expect [2] that

$$\frac{1}{\tau}\log\int_{-\infty}^{\infty}\mathrm{d}\hat{\mu}\int_{-\infty}^{\infty}\mathrm{d}\hat{\nu}\,\mathrm{e}^{\tau\Psi[\mu,\nu,\hat{\mu},\hat{\nu}]} = \mathrm{extr}_{\hat{\mu},\hat{\nu}}\,\Psi[\mu,\nu,\hat{\mu},\hat{\nu}]. \tag{32}$$

Now the equation $\frac{\partial\Psi}{\partial\hat{\mu}} = 0$, i.e. necessary condition for extremum, gives us

$$\mu = \frac{q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}}{1-q+q\,\mathrm{e}^{-\mathrm{i}\hat{\mu}}} \tag{33}$$

and solving above we obtain $\mathrm{i}\hat{\mu} = -\ln\left(\frac{\mu(q-1)}{q(\mu-1)}\right)$. The equation $\frac{\partial\Psi}{\partial\hat{\nu}} = 0$ gives us

$$\nu = \frac{\mu\,s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}}{1-s+s\,\mathrm{e}^{-\mathrm{i}\hat{\nu}}} \tag{34}$$

and solving above we obtain $\mathrm{i}\hat{\nu} = -\ln\left(\frac{(\nu/\mu)(s-1)}{s((\nu/\mu)-1)}\right)$. Hence the function (31) at the extremum is given by

$$\begin{aligned}
\tilde{\Psi}[\mu,\nu] &= \mu\ln\left(\frac{q\,(1-\mu)}{\mu\,(1-q)}\right) + \log\left(1-q+\frac{\mu\,(1-q)}{(1-\mu)}\right) \\
&\quad + \mu\left[(\nu/\mu)\ln\left(\frac{s\,(1-(\nu/\mu))}{(\nu/\mu)\,(1-s)}\right)\right. \\
&\quad \left. + \log\left(1-s+\frac{(\nu/\mu)\,(1-s)}{(1-(\nu/\mu))}\right)\right] \\
&= -\mu\log\left(\frac{\mu\,(1-q)}{q\,(1-\mu)}\right) + \log\left(\frac{1-q}{1-\mu}\right) \\
&\quad + \mu\left[-(\nu/\mu)\log\left(\frac{(\nu/\mu)\,(1-s)}{s\,(1-(\nu/\mu))}\right) + \log\left(\frac{1-s}{1-\nu}\right)\right] \\
&= -\left[\mathrm{D}(\mu||q) + \mu\,\mathrm{D}(\nu/\mu||s)\right],
\end{aligned} \tag{35}$$

where

$$\mathrm{D}(\mu||q) = \mu\log\left(\frac{\mu}{q}\right) + (1-\mu)\log\left(\frac{1-\mu}{1-q}\right) \tag{36}$$

is the Kullback-Leibler (KL) divergence which for $\mu, q \in [0,1]$ is 0 when $\mu = q$ and is positive semi-definite when $\mu \neq q$. Thus for $\tau \to \infty$ we have

$$\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{P}_\tau(\mu,\nu)\, f(\mu,\nu) = \frac{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]} f(\mu,\nu)}{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}} \tag{37}$$

which gives us

$$\mathrm{P}_\tau(\mu,\nu) = \frac{\mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}}{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}} \tag{38}$$

in this limit. From above, using the definition (24), we also obtain

$$\tilde{\mathrm{P}}_\tau(\mu,\nu) = \frac{\mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau \delta(\mu)\,\delta(\nu)}{1-(1-q)^\tau}. \tag{39}$$

- For $\epsilon > 0$ the event $s-\epsilon \le n/m \le s+\epsilon$, where $m \ge 1$, is equivalent to $(s-\epsilon)\,\mu \le \nu \le (s+\epsilon)\,\mu$, where $\mu > 0$. The probability of the latter can be estimated using (38), when $\tau \to \infty$, as follows

$$\begin{aligned}
&\mathrm{P}\left((s-\epsilon)\,\mu \le \nu \le (s+\epsilon)\,\mu\right) \\
&\quad = 1 - \mathrm{P}\left(\nu \le (s-\epsilon)\,\mu\right) - \mathrm{P}\left(\nu \ge (s+\epsilon)\,\mu\right) \\
&\quad = 1 - \frac{\int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}}{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}} \\
&\qquad - \frac{\int_0^1 \mathrm{d}\mu \int_{(s+\epsilon)\,\mu}^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}}{\int_0^1 \mathrm{d}\mu \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}}.
\end{aligned} \tag{40}$$

Let us define the difference $\Delta(\nu) = \nu - s$ and consider the integral

$$\begin{aligned}
\int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau\mu\,\mathrm{D}(\nu||s)} g(\nu) &= \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau\mu\,\mathrm{D}(s+\Delta(\nu)||s)} g(\nu) \\
&= \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau\mu\left[\frac{(\nu-s)^2}{2s(1-s)}+O(\Delta^3(\nu))\right]} g(\nu) \\
&\approx \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\frac{1}{2s(1-s)/\tau\mu}(\nu-s)^2} g(\nu).
\end{aligned} \tag{41}$$

where to obtain the last line we used ideas of from [3, 4]. Thus we have

$$\int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\tau\mu\,\mathrm{D}(\nu/\mu||s)} g(\nu) \approx \int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\frac{1}{2\sigma^2}\left(\frac{\nu}{\mu}-s\right)^2} g(\nu), \tag{42}$$

where $\sigma^2(\mu) = s(1-s)/\tau\mu$, for some test function $g(\nu)$. Now using above approximation in (38) allows us to compute the integrals

$$\begin{aligned}
&\int_0^\mu \mathrm{d}\nu\, \mathrm{e}^{-\frac{1}{2\sigma^2(\mu)}\left(\frac{\nu}{\mu}-s\right)^2} \\
&\qquad = \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2}\left[\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(\mu)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(\mu)}}\right)\right],
\end{aligned} \tag{43}$$

$$\begin{aligned}
&\int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \mathrm{e}^{-\frac{1}{2\sigma^2(\mu)}\left(\frac{\nu}{\mu}-s\right)^2} \\
&\qquad = \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2}\left[\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(\mu)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(\mu)}}\right)\right]
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\int_{(s+\epsilon)\,\mu}^\mu \mathrm{d}\nu\, \mathrm{e}^{-\frac{1}{2\sigma^2(\mu)}\left(\frac{\nu}{\mu}-s\right)^2} \\
&\qquad = \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2}\left[\mathrm{erf}\left(\frac{1-s}{\sqrt{2\sigma^2(\mu)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(\mu)}}\right)\right]
\end{aligned} \tag{45}$$

which can be used to approximate the probability

$$\begin{aligned}
&\mathrm{P}\left(\nu \leq (s-\epsilon)\,\mu\right) \\
&\approx \frac{\int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-\tau \mathrm{D}(\mu||q)} \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2} \left[\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(\mu)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(\mu)}}\right)\right]}{\int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-\tau \mathrm{D}(\mu||q)} \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2} \left[\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(\mu)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(\mu)}}\right)\right]} \\
&\approx \frac{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)},
\end{aligned} \tag{46}$$

where in above we assumed that for *large* τ the random variable μ can be replaced by its average q. Using the same steps we also obtain

$$\begin{aligned}
&\mathrm{P}\left(\nu \geq (s+\epsilon)\,\mu\right) \\
&\approx \frac{\int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-\tau \mathrm{D}(\mu||q)} \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2} \left[\mathrm{erf}\left(\frac{1-s}{\sqrt{2\sigma^2(\mu)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(\mu)}}\right)\right]}{\int_0^1 \mathrm{d}\mu\, \mathrm{e}^{-\tau \mathrm{D}(\mu||q)} \frac{\mu\sqrt{2\pi\sigma^2(\mu)}}{2} \left[\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(\mu)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(\mu)}}\right)\right]} \\
&\approx \frac{\mathrm{erf}\left(\frac{1-s}{\sqrt{2\sigma^2(q)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)}.
\end{aligned} \tag{47}$$

Finally, combining all of the above results we obtain the approximation

$$\begin{aligned}
&\mathrm{P}\left((s-\epsilon)\,\mu \leq \nu \leq (s+\epsilon)\,\mu\right) \\
&\approx 1 - \frac{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)} \\
&\quad - \frac{\mathrm{erf}\left(\frac{1-s}{\sqrt{2\sigma^2(q)}}\right) - \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \mathrm{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)} \\
&= \frac{2\,\mathrm{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\mathrm{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \mathrm{erf}\left(\frac{1-s}{\sqrt{2\sigma^2(q)}}\right)},
\end{aligned} \tag{48}$$

where $\sigma^2(q) = s(1-s)/\tau q$.

- We note that if we are interested only in the probability $\mathrm{P}\left(\nu \geq (s-\epsilon)\,\mu\right) = 1 - \mathrm{P}\left(\nu \leq (s-\epsilon)\,\mu\right)$, i.e. the probability that the estimator of s, ν/μ, is greater than $s-\epsilon$, then this probability is approximated by

$$\begin{aligned}
&1 - \mathrm{P}\left(\nu \leq (s-\epsilon)\,\mu\right) \\
&\quad = 1 - \frac{\int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q) + \mu\,\mathrm{D}(\nu/\mu||s)]}}{\int_0^1 \mathrm{d}\mu \int_0^{\mu} \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q) + \mu\,\mathrm{D}(\nu/\mu||s)]}}
\end{aligned}$$

$$\approx 1 - \frac{\operatorname{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) - \operatorname{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\operatorname{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \operatorname{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)}, \tag{49}$$

where $\sigma^2(q) = s(1-s)/\tau q$, and for

$$\begin{aligned}
&1 - \tilde{\mathrm{P}}\left(\nu \le (s-\epsilon)\,\mu\right) \\
&\quad = 1 - \int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \tilde{\mathrm{P}}_\tau(\mu,\nu) \\
&\quad = 1 - \int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \frac{\mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau \delta(\mu)\,\delta(\nu)}{1-(1-q)^\tau} \\
&\quad = 1 - \frac{\int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau}{1-(1-q)^\tau} \\
&\quad = 1 - \frac{\frac{\int_0^1 \mathrm{d}\mu \int_0^{(s-\epsilon)\,\mu} \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}}{\int_0^1 \mathrm{d}\mu \int_0^{\mu} \mathrm{d}\nu\, \mathrm{e}^{-\tau[\mathrm{D}(\mu||q)+\mu\,\mathrm{D}(\nu/\mu||s)]}} - (1-q)^\tau}{1-(1-q)^\tau} \\
&\quad \approx 1 - \frac{\frac{\operatorname{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) - \operatorname{erf}\left(\frac{\epsilon}{\sqrt{2\sigma^2(q)}}\right)}{\operatorname{erf}\left(\frac{s}{\sqrt{2\sigma^2(q)}}\right) + \operatorname{erf}\left(\frac{(1-s)}{\sqrt{2\sigma^2(q)}}\right)} - (1-q)^\tau}{1-(1-q)^\tau},
\end{aligned} \tag{50}$$

$$\tilde{\mathrm{P}}_\tau(\mu,\nu) = \frac{\mathrm{P}_\tau(\mu,\nu) - (1-q)^\tau \delta(\mu)\,\delta(\nu)}{1-(1-q)^\tau}. \tag{51}$$

Appendix A. Bounds on tails of the binomial distribution

Appendix A.1. Upper tail

The probability

$$\mathrm{P}(n \ge \lfloor (s+\epsilon)m \rfloor + 1 | m) = \sum_{n=\lfloor (s+\epsilon)m \rfloor + 1}^{m} \mathrm{P}\left(n|m\right) \tag{A.1}$$

can be bounded from the above

$$\mathrm{P}(n \ge \lfloor (s+\epsilon)m \rfloor + 1 | m) \le \mathrm{e}^{-m\,\mathrm{D}\left(\frac{\lfloor (s+\epsilon)m \rfloor + 1}{m} || s\right)} \tag{A.2}$$

and from the below

$$\mathrm{P}(n \ge \lfloor (s+\epsilon)m \rfloor + 1 | m) \ge \frac{\mathrm{e}^{-m\mathrm{D}\left(\frac{\lfloor (s+\epsilon)m \rfloor + 1}{m} || s\right)}}{\sqrt{8m\frac{\lfloor (s+\epsilon)m \rfloor + 1}{m}\left(1 - \frac{\lfloor (s+\epsilon)m \rfloor + 1}{m}\right)}} \tag{A.3}$$

for $s < \frac{\lfloor (s+\epsilon)m \rfloor + 1}{m} < 1$ by the *Lemma* 4.7.2 in [1]. In the case when $m = 1$ we have $\frac{\lfloor (s+\epsilon)m \rfloor + 1}{m} = 1$ if $0 < s + \epsilon < 1$ but here the probability

$$\mathrm{P}(n \ge \lfloor (s+\epsilon)m \rfloor + 1 | m) = \mathrm{P}\left(1|1\right) = s. \tag{A.4}$$

We note that $s + \epsilon < \frac{\lfloor (s+\epsilon)m \rfloor + 1}{m} \le s + \epsilon + 1/m$ and $\mathrm{D}\left(s+\epsilon||s\right)$ is monotonic increasing function of ϵ when $2s - 1 < \epsilon \le 1 - s$ which gives us

$$\mathrm{P}(n \ge \lfloor (s+\epsilon)m \rfloor + 1 | m) \le \mathrm{e}^{-m\,\mathrm{D}(s+\epsilon||s)} \tag{A.5}$$

and

$$\mathrm{P}(n \geq \lfloor (s+\epsilon)m \rfloor + 1 | m) \geq \frac{\mathrm{e}^{-m\,\mathrm{D}(s+\epsilon+1/m||s)}}{\sqrt{8m\frac{\lfloor (s+\epsilon)m\rfloor+1}{m}\left(1-\frac{\lfloor (s+\epsilon)m\rfloor+1}{m}\right)}}$$
$$\geq \frac{\mathrm{e}^{-m\,\mathrm{D}(s+\epsilon+1/2||s)}}{\sqrt{8\tau\frac{\lfloor (s+\epsilon)\tau\rfloor+1}{\tau}\left(1-\frac{\lfloor (s+\epsilon)\tau\rfloor+1}{\tau}\right)}} \quad \text{(A.6)}$$

where to obtain the second inequality in above we assumed that $2 \leq m \leq \tau$ and $s+\epsilon \leq 1/2$.

Appendix A.2. Lower tail

The probability

$$\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) = \sum_{n=0}^{\lfloor (s-\epsilon)m \rfloor} \mathrm{P}\left(n|m\right) \quad \text{(A.7)}$$

can be estimated as follows

$$\begin{aligned}\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) &= \mathrm{P}(\mathrm{e}^{-\lambda n} \geq \mathrm{e}^{-\lambda \lfloor (s-\epsilon)m \rfloor} | m) \\ &\leq \sum_{n=0}^{m} \mathrm{P}\left(n|m\right) \mathrm{e}^{-\lambda n} \mathrm{e}^{\lambda \lfloor (s-\epsilon)m \rfloor} \\ &= (1-s+s\,\mathrm{e}^{-\lambda})^m \mathrm{e}^{\lambda \lfloor (s-\epsilon)m \rfloor} \\ &= \mathrm{e}^{-m\,\phi(\lambda)}, \end{aligned} \quad \text{(A.8)}$$

where

$$\phi(\lambda) = -\log(1-s+s\,\mathrm{e}^{-\lambda}) - \lambda \frac{\lfloor (s-\epsilon)m \rfloor}{m}. \quad \text{(A.9)}$$

Now the function $\phi(\lambda)$ is *concave* and has a maximum at $\lambda = \log\left(\frac{s(1-\frac{\lfloor (s-\epsilon)m\rfloor}{m})}{\frac{\lfloor (s-\epsilon)m\rfloor}{m}(1-s)}\right)$. The latter is positive when $\frac{\lfloor (s-\epsilon)m \rfloor}{m} < s$, so can be used to optimise the upper bound $\mathrm{e}^{-m\,\phi(\lambda)}$ giving us

$$\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \leq \mathrm{e}^{-m\,\mathrm{D}(\frac{\lfloor (s-\epsilon)m \rfloor}{m}||s)} \quad \text{(A.10)}$$

for $\frac{\lfloor (s-\epsilon)m \rfloor}{m} < s$. Furthermore, we have $\frac{\lfloor (s-\epsilon)m \rfloor}{m} < s-\epsilon < s$ and hence

$$\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \leq \mathrm{e}^{-m\,\mathrm{D}(s-\epsilon||s)}. \quad \text{(A.11)}$$

Let us now obtain the lower bound for (A.7). This can be done as follows

$$\begin{aligned}&\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \\ &= \sum_{n=0}^{\lfloor (s-\epsilon)m \rfloor} \mathrm{P}\left(n|m\right) \\ &= \mathbb{1}\left[(s-\epsilon)\,m < 1\right] \mathrm{P}\left(0|m\right) + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \sum_{n=0}^{\lfloor (s-\epsilon)m \rfloor} \mathrm{P}\left(n|m\right) \\ &= \mathbb{1}\left[(s-\epsilon)\,m < 1\right] (1-s)^m + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \sum_{n=0}^{\lfloor (s-\epsilon)m \rfloor} \mathrm{P}\left(n|m\right) \end{aligned}$$

$$
\begin{aligned}
&\geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \mathrm{P}\left(\lfloor (s-\epsilon)m \rfloor | m\right) \\
&= \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m \\
&\quad + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \binom{m}{\lfloor (s-\epsilon)m \rfloor} s^{\lfloor (s-\epsilon)m \rfloor} (1-s)^{m-\lfloor (s-\epsilon)m \rfloor} \\
&= \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m \\
&\quad + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \binom{m}{\lfloor (s-\epsilon)m \rfloor} \mathrm{e}^{-m\, \mathrm{H}(\frac{\lfloor (s-\epsilon)m \rfloor}{m})} \mathrm{e}^{-m\, \mathrm{D}(\frac{\lfloor (s-\epsilon)m \rfloor}{m} || s)} \\
&\geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m \\
&\qquad + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \frac{\mathrm{e}^{m\, \mathrm{H}(\frac{\lfloor (s-\epsilon)m \rfloor}{m})}}{\sqrt{8m \frac{\lfloor (s-\epsilon)m \rfloor}{m} \left(1 - \frac{\lfloor (s-\epsilon)m \rfloor}{m}\right)}} \\
&\qquad\quad \times \mathrm{e}^{-m\, \mathrm{H}(\frac{\lfloor (s-\epsilon)m \rfloor}{m})} \mathrm{e}^{-m\, \mathrm{D}(\frac{\lfloor (s-\epsilon)m \rfloor}{m} || s)}
\end{aligned} \tag{A.12}
$$

and hence

$$
\begin{aligned}
&\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \\
&\qquad \geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m + \cdots \\
&\qquad\quad \cdots + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \frac{\mathrm{e}^{-m\, \mathrm{D}(\frac{\lfloor (s-\epsilon)m \rfloor}{m} || s)}}{\sqrt{8m \frac{\lfloor (s-\epsilon)m \rfloor}{m} \left(1 - \frac{\lfloor (s-\epsilon)m \rfloor}{m}\right)}} \\
&\qquad \geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m + \cdots \\
&\qquad\quad \cdots + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \frac{\mathrm{e}^{-m \max_{\tilde{m} \geq 1/(s-\epsilon)} \mathrm{D}(\frac{\lfloor (s-\epsilon)\tilde{m} \rfloor}{\tilde{m}} || s)}}{\sqrt{8\tau \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} \left(1 - \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau}\right)}}.
\end{aligned} \tag{A.13}
$$

We note that by using $\binom{m}{\lfloor (s-\epsilon)m \rfloor} \geq \frac{\mathrm{e}^{m\, \mathrm{H}(\frac{\lfloor (s-\epsilon)m \rfloor}{m})}}{m+1}$ we can obtain a less tight bound

$$
\begin{aligned}
&\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \\
&\quad \geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m + \cdots \\
&\qquad \cdots + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \frac{\mathrm{e}^{-m\, \mathrm{D}(\frac{\lfloor (s-\epsilon)m \rfloor}{m} || s)}}{m+1} \\
&\qquad \geq \mathbb{1}\left[(s-\epsilon)\, m < 1\right](1-s)^m + \cdots \\
&\qquad\quad \cdots + \mathbb{1}\left[(s-\epsilon)m \geq 1\right] \frac{\mathrm{e}^{-m \max_{\tilde{m} \geq 1/(s-\epsilon)} \mathrm{D}(\frac{\lfloor (s-\epsilon)\tilde{m} \rfloor}{\tilde{m}} || s)}}{\tau+1},
\end{aligned} \tag{A.14}
$$

where in above we assumed that $1 \leq m \leq \tau$.

Let us try to obtain a much tighter lower bound as follows. For $(s-\epsilon)m \geq 1$ we have

$$
\begin{aligned}
&\mathrm{P}(n \leq \lfloor (s-\epsilon)m \rfloor | m) \\
&\quad = \sum_{n=0}^{\lfloor (s-\epsilon)m \rfloor} \mathrm{P}\left(n | m\right) \\
&\quad \geq \frac{1 - c(Q(m), 1/r)/m}{1 - 1/r} \frac{1}{\sqrt{2\pi m\, Q(m)\left[1 - Q(m)\right]}} \mathrm{e}^{-m\, \mathrm{D}(Q(m) || s)}
\end{aligned} \tag{A.15}
$$

where $Q(m) = \frac{\lfloor (s-\epsilon)m \rfloor}{m}$

$$r = \frac{s\,[1 - Q(m)]}{Q(m)(1-s)} \tag{A.16}$$

$$c(a, r) = \frac{1}{a(1-a)} \left[1 + \frac{r(1+r)}{(1-r)^2}\right] \tag{A.17}$$

Appendix B. Results for $q = 1$

For $q = 1$ we have

$$\mathrm{P}\,(n, m|\tau, m \geq 1) \; = \mathrm{P}\,(n|\tau) \tag{B.1}$$

then

$$\begin{aligned}
\mathrm{P}(n \leq \lfloor (s-\epsilon)\tau \rfloor | m) &= \sum_{n=0}^{\lfloor (s-\epsilon)\tau \rfloor} \mathrm{P}\,(n|\tau) \\
&= \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\quad + \mathbb{1}[(s-\epsilon)\tau \geq 1] \sum_{n=0}^{\lfloor (s-\epsilon)\tau \rfloor} \mathrm{P}\,(n|\tau) \\
&\leq \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\quad + \mathbb{1}[(s-\epsilon)\tau \geq 1]\, \mathrm{e}^{-\tau\, \mathrm{D}\left(\frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} || s\right)} \\
&\leq \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\quad + \mathbb{1}[(s-\epsilon)\tau \geq 1]\, \mathrm{e}^{-\tau\, \mathrm{D}(s-\epsilon || s)}
\end{aligned} \tag{B.2}$$

and hence we obtain the following upper bound

$$\begin{aligned}
\mathrm{P}(n \leq \lfloor (s-\epsilon)\tau \rfloor | m) &\leq \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\quad + \mathbb{1}[(s-\epsilon)\tau \geq 1]\, \mathrm{e}^{-\tau\, \mathrm{D}(s-\epsilon || s)}.
\end{aligned} \tag{B.3}$$

Let us now derive the lower bound

$$\begin{aligned}
&\mathrm{P}(n \leq \lfloor (s-\epsilon)\tau \rfloor | m) \\
&\quad = \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\qquad + \mathbb{1}[(s-\epsilon)\tau \geq 1] \sum_{n=0}^{\lfloor (s-\epsilon)\tau \rfloor} \mathrm{P}\,(n|\tau) \\
&\quad \geq \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\qquad + \mathbb{1}[(s-\epsilon)\tau \geq 1] \frac{\mathrm{e}^{-\tau\, \mathrm{D}\left(\frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} || s\right)}}{\sqrt{8\tau \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} \left(1 - \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau}\right)}}
\end{aligned} \tag{B.4}$$

Now using above bounds we obtain the following

$$\begin{aligned}
&\mathrm{P}\,(n \geq \lfloor (s-\epsilon)\, m \rfloor + 1 | m \geq 1) \\
&\quad = 1 - \mathrm{P}\,(n \leq \lfloor (s-\epsilon)\, m \rfloor | m \geq 1) \\
&\quad \geq 1 - \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} \\
&\qquad - \mathbb{1}[(s-\epsilon)\tau \geq 1]\, \mathrm{e}^{-\tau\, \mathrm{D}(s-\epsilon || s)}
\end{aligned}$$

$$\leq 1 - \mathbb{1}[(s-\epsilon)\tau < 1](1-s)^{\tau} - \mathbb{1}[(s-\epsilon)\tau \geq 1] \frac{\mathrm{e}^{-\tau\, \mathrm{D}(\frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} || s)}}{\sqrt{8\tau \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau} \left(1 - \frac{\lfloor (s-\epsilon)\tau \rfloor}{\tau}\right)}}. \tag{B.5}$$

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